Grade Level/Course: Algebra 1
Lesson/Unit Plan Name: Geometric Sequences
Rationale/Lesson Abstract: What makes a sequence geometric? This characteristic is addressed in the definition of a geometric sequence and will help derive the recursive formula. Students will write the recursive and explicit formulas for geometric sequences.
Timeframe: 2 class periods
Common Core Standard F-BF.2: Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms. **Notes:** The Warm-Up is on page 12. A black-line master of Example 3 You Try is on provided on page 10 for duplication or use with a projection system. There are two forms of the recursive formula $a_n = a_{n-1} \cdot r$ and $a_n = r \cdot a_{n-1}$. These two forms are used interchangeably in this lesson.

Instructional Resources/Materials: Warm-Up, Black-line master, Example 4 visual aid, Index Cards (optional)

Lesson:

Think-Pair-Share: Describe the pattern in each sequence.

a) $\frac{3}{4}, \frac{3}{2}, 3, 6, 12, \ldots$

b) $-5, -3, -1, 1, 3, \ldots$

TPS Answers:

a) Each term is 2 times the previous term. (Also, the sequence is not arithmetic.)

b) Each term is 2 more than the previous term. (Also, the sequence is arithmetic.)

REVIEW from the Arithmetic Sequence Lesson:

A sequence is a list or an ordered arrangement of numbers, figures or objects. The members, which are also elements, are called the terms of the sequence. A general sequence can be written as

$$a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6, \ldots$$

where a_1 is the first term, a_2 is the second term, and so on. The n[th] term is denoted as a_n.

A geometric sequence is a list of numbers in which the ratio of any term to the previous term is constant. The constant ratio is called the common ratio is denoted by *r*.

$$r = \frac{a_n}{a_{n-1}}$$

Example 1: Determine if the sequence is geometric. Justify your answer.

Use the definition and check if all ratios $\frac{a_n}{a_{n-1}}$ are the same.

Sequence	$\frac{a_2}{a_1}$	$\frac{a_3}{a_2}$	$\frac{a_4}{a_3}$	$\frac{a_5}{a_4}$	Conclusion
$\frac{3}{4}, \frac{3}{2}, 3, 6, 12, \ldots$	$= \frac{3}{2} \div \frac{3}{4}$ $= \frac{3}{2}\left(\frac{2}{2}\right) \div \frac{3}{4}$ $= \frac{6}{4} \div \frac{3}{4}$ $= \frac{6 \div 3}{4 \div 4}$ $= 2$	$= \frac{3}{1} \div \frac{3}{2}$ $= \frac{3}{1}\left(\frac{2}{2}\right) \div \frac{3}{2}$ $= \frac{6}{2} \div \frac{3}{2}$ $= \frac{6 \div 3}{2 \div 2}$ $= 2$	$= \frac{6}{3}$ $= 2$	$= \frac{12}{6}$ $= 2$	Since all the ratios are constant (constantly 2), the sequence is geometric and the common ratio is $\frac{a_n}{a_{n-1}} = 2$.

Try

$-5, -3, -1, 1, 3, ..$	$= \frac{-3}{-5}$ $= \frac{3}{5}$	$= \frac{-1}{-3}$ $= \frac{1}{3}$	$= \frac{1}{-1}$ $= -1$	$= \frac{3}{1}$ $= 3$	Since all the ratios $\frac{a_n}{a_{n-1}}$ are different and NOT the same constant, then the sequence is NOT geometric.

Another way to represent the ratio symbolically is to use arrows showing the multiplication from one term to the next.

$$\frac{3}{4},\quad \frac{3}{2},\quad 3,\quad 6,\quad 12, \ldots$$

$$\bullet\frac{2}{1}\quad \bullet\frac{2}{1}\quad \bullet\frac{2}{1}\quad \bullet\frac{2}{1}$$

$$-5,\quad -3,\quad -1,\quad 1,\quad 3, \ldots$$

$$\bullet\frac{3}{5}\quad \bullet\frac{1}{3}\quad \bullet -1\quad \bullet 3$$

Think-Pair-Share: Explain to your partner what the equation $r = \frac{a_n}{a_{n-1}}$ is used for and how to use it?

Can we rewrite this equation in another form?

<u>Derive the Recursive Formula of a Geometric Sequence</u>

Solve $r = \frac{a_n}{a_{n-1}}$ for a_n:

$$\boxed{\begin{aligned} r &= \frac{a_n}{a_{n-1}} \\ a_{n-1} \cdot r &= a_{n-1} \cdot \frac{a_n}{a_{n-1}} \\ a_{n-1} \cdot r &= a_n \\ a_n &= a_{n-1} \cdot r \end{aligned}}$$

$\therefore r = \frac{a_n}{a_{n-1}}$, $a_n = a_{n-1} \cdot r$ (or $a_n = r \cdot a_{n-1}$) are equivalent equations.

The equation $a_n = a_{n-1} \cdot r$ is called the <u>recursive formula</u> of a geometric sequence. It defines the next term as the previous term times the common ratio. It can be used to generate the terms of a geometric sequence one term at a time.

Example 2: Given the geometric sequence $4, 40, 400, 4000, \ldots$

a) Find the next term.

Since we know the sequence is geometric, there is a common ratio. What is it?

Use $r = \frac{a_n}{a_{n-1}}$: $r = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3}$. Using $r = \frac{a_3}{a_2} = \frac{400}{40}$ $\therefore r = 10$.

Method 1 Since we know the fourth term and the common ratio, we can use the recursive formula to find the fifth term. Use $a_n = a_{n-1} \cdot r$. Substitute and simplify. $a_5 = a_{5-1} \cdot 10$ $= a_4 \cdot 10$ $= 4000 \cdot 10$ $= 40000$	**Method 2** Rewrite each term in terms of the first term and the common ratio. $a_1 = 4$ $a_2 = 4 \cdot 10^1$ $a_3 = 4 \cdot 10^2$ $a_4 = 4 \cdot 10^3$ $a_5 = 4 \cdot 10^?$ $a_5 = 4 \cdot 10^4$
Recursive Formula: Use the equation derived on the previous page. The recursive formula is used to find the next term in the sequence by multiplying the previous term by the common ratio. $a_n = a_{n-1} \cdot r$ $a_n = a_{n-1} \cdot 10$ The recursive formula is $a_n = 10 \cdot a_{n-1}$, $a_1 = 4$ *Remember there are two parts to this formula.	**Explicit Formula:** Use the pattern to find any term. The explicit formula is used to find any term in the sequence, without knowing the previous term. $a_5 = 4 \cdot 10^4$ $\ldots$ $a_n = 4 \cdot 10^{n-1}$ *Use more terms if needed to get students to see this pattern.

Think-Pair-Share: Which formula(s) can be used to find a_{20}? Which formula would be most efficient? Justify your answer.

Both formulas can be used to find the 20th term. Method 1 is not very time efficient as you would need to find all the terms leading up to a_{20}. Method 2 is the most direct approach since you only need to know the value of *n*, which in this case is 20.

b) Find a_{20}.

If $a_n = 4 \cdot 10^{n-1}$, then $a_{20} = 4 \cdot 10^{20-1}$. Therefore, $a_{20} = 4 \cdot 10^{19}$.

Discuss: Should we evaluate $a_{20} = 4 \cdot 10^{19}$ or leave it as is?

Refer back to the definition of a geometric sequence and generalize :

> A geometric sequence can be written as $a_1,\ a_1r,\ a_1r^2,\ a_1r^3,\ a_1r^4,\ a_1r^5, \ldots,\ a_1r^{n-1},\ a_1r^n, \ldots$
>
> where a_1 is the first term, or initial condition, and r is the common ratio.

Like arithmetic sequences, geometric sequences also have recursive and explicit formulas.

The formulas for arithmetic sequences are provided for review and application.

Type of Sequence	Recursive Formula (or rule)	Explicit Formula (or rule)
Arithmetic The common difference is $d = a_n - a_{n-1}$.	$a_n = a_{n-1} + d$ where a_1 is given	$a_n = a_1 + (n-1)d$
Geometric The common ratio is $r = \frac{a_n}{a_{n-1}}$.	$a_n = a_{n-1} \cdot r$ where a_1 is given	$a_n = a_1 \cdot r^{n-1}$

<u>**Example 3a:**</u> Write the recursive and explicit formulas for the sequence $-64, 16, -4, 1, \ldots$

Think-Pair: What kind of sequence is $-64, 16, -4, 1, \ldots$? Which formulas do we use?

The sequence is geometric, so use the geometric formulas.

Common Ratio	Recursive Formula (or rule)	Explicit Formula (or rule)
$r = \frac{a_4}{a_3}$ $r = \frac{1}{-4}$ or $-\frac{1}{4}$	Use $a_n = a_{n-1} \cdot r$ and replace the r value and state the first term. $a_n = a_{n-1} \cdot \left(-\frac{1}{4}\right)$ where $a_1 = -64$	Use $a_n = a_1 \cdot r^{n-1}$ and replace the r value and the first term. $a_n = -64 \cdot \left(-\frac{1}{4}\right)^{n-1}$

Example 3b (optional): Use both formulas to find a_6.

Recursive Formula (or rule)	Explicit Formula (or rule)
Use $a_n = a_{n-1} \cdot \left(-\frac{1}{4}\right)$ where $a_1 = -64$ and $n = 6$. Identify the terms up to a_6: $a_1 = -64$ $a_2 = 16$ $a_3 = -4$ $a_4 = 1$ $a_5 = ?$ $a_6 = ?$ 1st: Find a_5 $a_5 = a_4 \cdot \left(-\frac{1}{4}\right)$ $a_5 = 1 \cdot \left(-\frac{1}{4}\right)$ $a_5 = -\frac{1}{4}$ 2nd: Find a_6 $a_6 = a_5 \cdot \left(-\frac{1}{4}\right)$ $a_6 = -\frac{1}{4} \cdot \left(-\frac{1}{4}\right)$ $a_6 = \frac{1}{16}$	Use $a_n = -64 \cdot \left(-\frac{1}{4}\right)^{n-1}$ where $n = 6$. $a_6 = -64 \cdot \left(-\frac{1}{4}\right)^{6-1}$ $a_6 = -64 \cdot \left(-\frac{1}{4}\right)^{5}$ $a_6 = -1 \cdot 2^6 \cdot \frac{(-1)^5}{4^5}$ $a_6 = \frac{-1 \cdot 2^6}{1} \cdot \frac{-1}{(2^2)^5}$ $a_6 = \frac{2^6}{2^{10}}$ $a_6 = \frac{1}{2^4}$ $a_6 = \frac{1}{16}$

TRY (with Solutions): Decide whether the sequence is arithmetic, geometric, or neither. Find the next term. Then write the recursive and explicit formulas.

	Sequence	Next term	Common Ratio or Difference	Recursive Formula (or rule)	Explicit Formula (or rule)
Partner A	$4, -12, 36, -108, \ldots$ (Geometric)	$a_5 = 324$	$r = -3$	$a_n = a_{n-1} \cdot (-3)$ where $a_1 = 4$	$a_n = 4 \cdot (-3)^{n-1}$
Partner B	$\frac{3}{4}, \frac{3}{2}, 3, 6, 12, \ldots$ (Geometric)	$a_6 = 24$	$r = 2$	$a_n = a_{n-1} \cdot (2)$ where $a_1 = \frac{3}{4}$	$a_n = \frac{3}{4} \cdot (2)^{n-1}$
	$-5, -3, -1, 1, 3, \ldots$ (Arithmetic)	$a_6 = 5$	$d = 2$	$a_n = a_{n-1} + 2$ where $a_1 = -5$	$a_n = -5 + (n-1)(2)$ or $a_n = 2n - 7$
Teacher's Choice					

Example 4: Suppose you drop a ball from a height of 100 cm.
It bounces back to 80% of its previous height.

a) About how high will the ball go after its fifth bounce?

Initial height of ball: 100 cm

After first bounce: 80% of 100 cm 0.80(100 cm) = 80 cm

After 2nd bounce: 80% of 80 cm 0.80(80 cm) = 64 cm

After 3rd bounce: 80% of <u>64 cm</u> 0.80(<u>64 cm</u>) = <u>51.2 cm</u>

After 4th bounce: 80% of <u>51.2 cm</u> 0.80(<u>51.2 cm</u>) = <u>40.96 cm</u>

After 5th bounce: 80% of <u>40.96 cm</u> 0.80(<u>40.96 cm</u>) = <u>32.768 cm</u>

> <u>Teacher Note:</u> Omit the underlined values and prompt students to identify the correct value that belongs in each blank.

Therefore, the ball will rebound about <u>32.8 cm</u> after the fifth bounce**.**

Think-Pair: What kind of sequence is created by these heights? How do you know?
Explain what the 100 cm and the 80% represent.

b) Write the recursive and explicit formulas for the geometric sequence generated by these heights.

To write both formulas, identify the common ratio and the first term: $r = 0.8$ and $a_1 = 100$

Recursive Formula: $a_n = (0.8)a_{n-1}$ where $a_1 = 100$

Explicit Formula: $a_n = 100 \cdot (0.8)^{n-1}$

TRY: *Model the situation below using a recursive and explicit formula.*

At the beginning of an experiment, there are 100 bacteria colonies.

The number of colonies doubles each hour.

*Note: This problem is from the warm-up.

Recursive Formula: $a_n = 2 \cdot a_{n-1}$ where $a_1 = 100$

Explicit Formula: $a_n = 100 \cdot (2)^{n-1}$

Think-Pair-Share/Discussion Questions:

Why is it necessary to identify the first term in the recursive formulas?

If the first term is not identified, then the formula represents any sequence that has the same common ratio. For example, $a_n = a_{n-1} \cdot (-3)$ represents the sequences $-1, 3, -9, 27, \ldots$ and $\frac{2}{3}, -2, 6, -18, \ldots$.

How do geometric sequences with a positive common ratio compare to geometric sequences with a negative common ratio?

The terms of a geometric sequence with a positive common ratio are the same sign (all positive or all negative), whereas the terms of a geometric sequence with a negative common ratio alternate signs.

How are geometric sequences similar to exponential functions?

The explicit formulas of geometric sequences are exponential functions.

How is a geometric sequence different from an exponential function?

An exponential function is continuous and the domain is all real numbers. A geometric sequence is a collection of points that are not connected which make it not continuous and the domain is all natural numbers $\{1, 2, 3, 4, \ldots, n\}$.

Is a geometric sequence a function?

Yes, a geometric sequence is a function whose domain is all natural numbers $\{1, 2, 3, 4, \ldots, n\}$.

Therefore, the explicit formula of a geometric sequence can be written in function notation.

Arithmetic and Geometric Sequences and Function Notation:

Sequence	**Recursive Formula in Function Notation**	**Explicit Formula in Function Notation**
Arithmetic	$f(n) = f(n-1) + d$ with $f(1)$ given	$f(n) = f(1) + (n-1)d$
Geometric	$f(n) = f(n-1) \cdot r$ with $f(1)$ given	$f(n) = f(1) \cdot r^{n-1}$

Exit Ticket Options:

Extension Activity/Option #1:

1. Each student creates their own geometric sequence and writes it on one side of an index card.

2. Each student writes the recursive and explicit formulas for their sequence on the other side of card.

3. Students write their names on one side of the index card and submit them to the teacher.

4. Teacher checks cards for accuracy.

5. Teacher returns cards to students on a different day so they can participate in the Quiz-Quiz-Trade Activity.

- Teacher tells students to stand up and pair up.
- Partner A quizzes B.
- Partner B answers.
- Partner A praises or coaches.
- Partners switch roles.
- Partners trade cards.
- After teacher specified time, repeat the steps above with a new partner.

Option #2: Determine if each statement is true or false.

1) The sequence $2, 4, 6, 8, 10, \ldots$ is geometric. ◯ True ◯ False

2) The sequence $2, 4, 8, 16, \ldots$ is geometric. ◯ True ◯ False

3) The recursive formula $a_n = \frac{1}{2} \cdot a_{n-1},\ a_1 = 80$ represents the sequence $80, 40, 20, 10, 5, \ldots$ ◯ True ◯ False

4) The explicit formula $a_n = 3 \cdot (-2)^{n-1}$ represents the sequence $-3, 6, -12, 24, \ldots$ ◯ True ◯ False

5) The recursive formula $a_n = (-1) \cdot a_{n-1},\ a_1 = 4$ and the explicit formula $a_n = 4(-1)^{n-1}$ represent different sequences. ◯ True ◯ False

ANSWERS: F, T, T, F, F

Type of Sequence	Recursive Formula (or rule)	Explicit Formula (or rule)
Arithmetic Have common difference $d = a_n - a_{n-1}$.	$a_n = a_{n-1} + d$ where a_1 is given	$a_n = a_1 + (n-1)d$
Geometric Have common ratio $r = \frac{a_n}{a_{n-1}}$.	$a_n = a_{n-1} \cdot r$ where a_1 is given	$a_n = a_1 \cdot r^{n-1}$

Sequence	Next term	Common Ratio or Difference	Recursive Formula (or rule)	Explicit Formula (or rule)
$4, -12, 36, -108, \ldots$ Circle Sequence Type: Geometric Arithmetic Neither				
$-5, -3, -1, 1, 3, \ldots$ Circle Sequence Type: Geometric Arithmetic Neither				
$\frac{3}{4}, \frac{3}{2}, 3, 6, 12, \ldots$ Circle Sequence Type: Geometric Arithmetic Neither				
Circle Sequence Type: Geometric Arithmetic Neither				

Example 4 Visuals

Warm-Up

CCSS: F-LE.2

Use the graph below to determine if each statement is true or false.

A) $f(x) = 2x + 2$ ○True ○False

B) $f(x) = 2^x + 2$ ○True ○False

C) $f(x) = 2 \cdot 2^x$ ○True ○False

D) $f(x) = 2^{x+1}$ ○True ○False

E) $f(x) \geq 0$ ○True ○False

Find the value of $f(3)$.

CCSS: F-BF.2

Given the arithmetic sequence $21, 18, 15, 12, \ldots$:

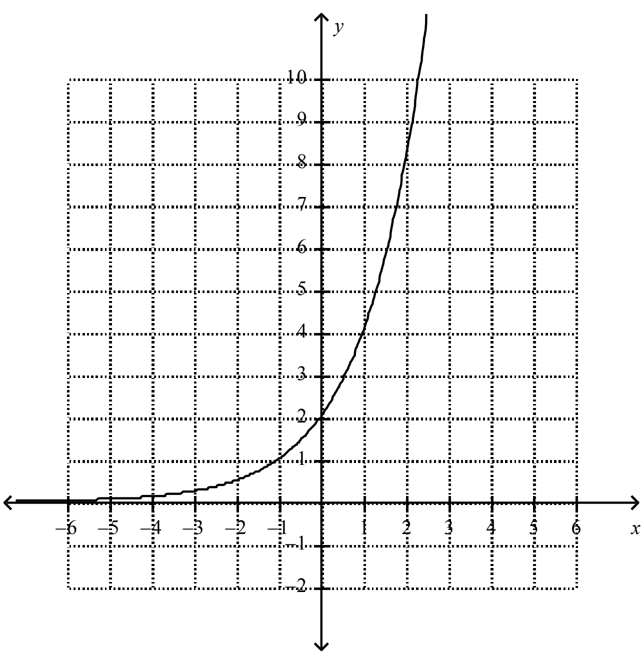
a) Describe why the sequence is arithmetic.

b) Identify the common difference.

c) Write the recursive formula.

d) Write the explicit formula.

CCSS: F-LE.2 continued



Current:

At the beginning of an experiment, there are 100 bacteria colonies. The number of colonies doubles each hour and is recorded. Complete the table.

Reading	Number of Colonies
1	100
2	
3	
4	
5	
n	
10	

Warm-Up Solutions

CCSS: F-LE.2

Use the graph below to determine if each statement is true or false.

A) $f(x) = 2x + 2$ — ○ True ● False

B) $f(x) = 2^x + 2$ — ○ True ● False

C) $f(x) = 2 \cdot 2^x$ — ● True ○ False

D) $f(x) = 2^{x+1}$ — ● True ○ False

E) $f(x) \geq 0$ — ● True ○ False

$f(3) = 16$

CCSS: F-BF.2

Given the arithmetic sequence $21, 18, 15, 12, \ldots$:

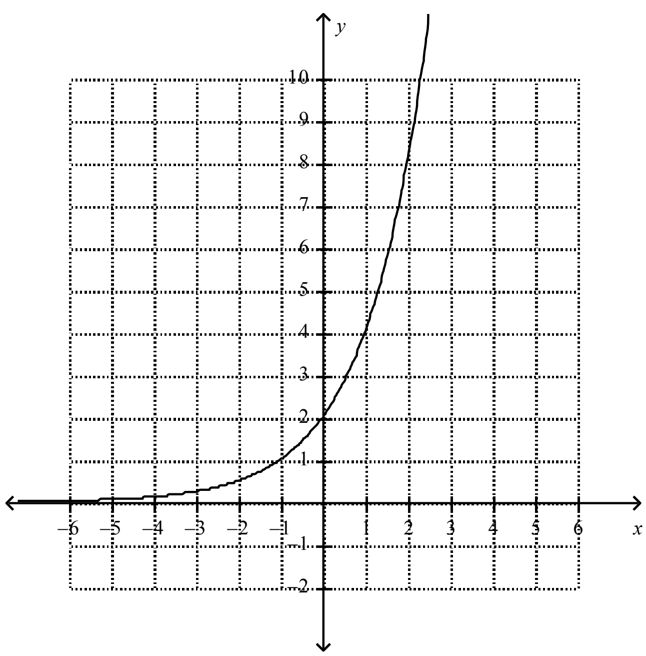
a) The sequence is arithmetic because there is a common difference between consecutive terms.

b) The common difference is -3.

c) $a_n = a_{n-1} - 3$ where $a_1 = 21$

d) $a_n = 21 - 3(n-1)$ or $a_n = -3n + 24$

CCSS: F-LE.2 continued



Current:

At the beginning of an experiment, there are 100 bacteria colonies. The number of colonies doubles each hour and is recorded. Complete the table.

Reading	Number of Colonies
1	100
2	$100 \cdot 2 = 200$
3	$100 \cdot 2^2 = 400$
4	$100 \cdot 2^3 = 800$
5	$100 \cdot 2^4 = 1600$
n	$100 \cdot 2^{n-1}$
10	$100 \cdot 2^9$